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LETTER TO THE EDITOR

On the connection between directed percolation and directed polymers

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Abstract. We consider the scaling behaviour of the directed paths of minimum energy on a lattice to each bond of which energy 0 or 1 is randomly assigned. In $1+1$ dimensions, it is shown by numerical transfer-matrix calculations that in all the regions below the directed-percolation threshold, these paths still scale with the directed-percolation exponent. At the same time, degeneracy of these paths at, and below, the threshold is found to be very different. The problem of stability of the directed-percolation ground state and the case when the energy on each bond can take randomly $q > 2$ different values are also considered.

The directed-polymer problem has recently attracted considerable attention [1, 2]. In continuous formulation, the problem is described by the imaginary-time Schrödinger equation

$$T \frac{\partial \varphi}{\partial t} = T^2 \Delta \varphi + V(x, t) \varphi \quad (1)$$

where $V(x, t)$ is a random potential and T is a constant (usually interpreted as temperature). In lattice formulation, $\varphi(x, t)$ can be considered as the sum over all directed paths on a lattice (figure 1), to each bond of which an energy $V(x, t)$ is randomly assigned. Each path is weighted with a corresponding Boltzmann factor $\exp(-V/T)$, where V is the sum of $V(x, t)$ along a given path $x(t)$. Consequently, to find the ground state (the $T \rightarrow 0$ limit) for the directed polymer, one should find the best energy paths on the lattice. Usually, the random energy $V(x, t)$ is assumed to have a continuous distribution. The case of a discrete-valued $V(x, t)$ has been studied much less, although it deserves special attention from the point of view of possible applications of the directed-polymer problem. Indeed, it was mentioned [3], that among other applications, equation (1) can be treated as a problem of the variation of population of some living beings, due to variations of life conditions $V(x, t)$. It is also natural to try to use this equation to describe the diffusion–reaction processes, like the so-called Schlogl’s first model [4]. In this case, to each reaction event, say to duplication or death of an individual, we assign a suitable value of the energy $V(x, t)$. However, in this case, both $\varphi(x, t)$ and $V(x, t)$ should be discrete. It is important to verify whether this discreteness will alter the directed-polymer universality class or not.

In order to test this, we study in this paper the scaling properties of the best energy paths on a lattice with discrete-valued random energies $V(x, t)$ in $1+1$ dimensions by the numerical transfer-matrix method. In the simplest case, the energies $V(x, t)$ can take

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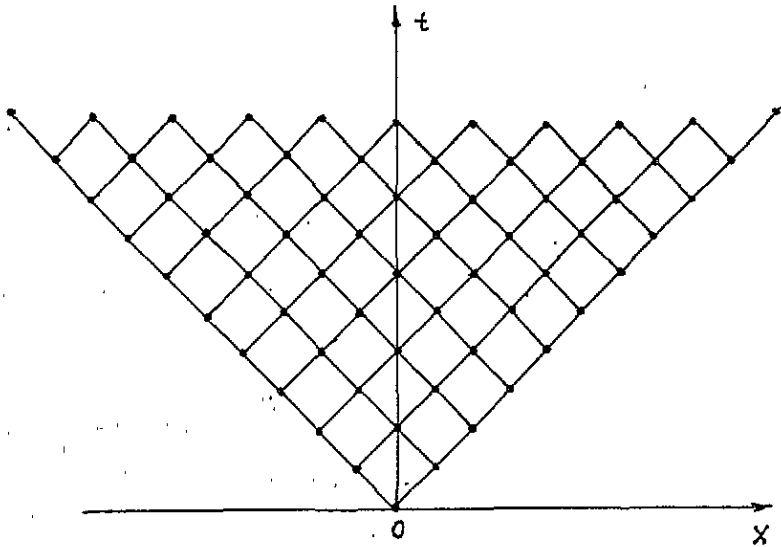


Figure 1. Square lattice in the diagonal direction, on which the numerical treatment was performed. Energy 0 or 1 is randomly assigned to each bond.

only two values, say 0 and 1. For one special concentration $p = p_c$ of the bonds with energy 0, which is the concentration of the directed-percolation threshold, the answer can be understood without any calculations. Indeed, at the directed-percolation threshold, the directed-percolation path will be the best energy path and, therefore, the exponent for the average transverse displacement of the best energy path will be that of the directed percolation [5]. This result indeed corresponds to the known answer for the Schlogl's model at criticality and differs from the usual answer for the directed-polymer problem with continuous random energy distribution [1, 2]. Recently, it was shown [5] that, above the threshold concentration, the usual directed-polymer exponent is restored. We investigate here the region below the threshold. We consider also the question of stability of the computed ground state and the case when the $V(x, t)$ can take $q > 2$ different values.

In order to test our program, we investigate first the best energy paths at the directed-percolation threshold, i.e. $p = p_c$. (For the random bond problem, it is known that $p_c = 0.6445$ [6]). The typical shape of the best energy-paths cluster (which is actually the directed-percolation cluster) is shown in figure 2(a). The cluster looks similar to that in [7]. (The only difference is that the 'death branches', i.e. those paths which did not reach the upper bound of the figure, are not shown in our case). The scaling of the position of the path furthest from the origin is shown in figure 3. In our calculations, the perpendicular size of the system was chosen to be $L = 1000$, and the data was averaged over 2000 or more realizations of $V(x, t)$. The exponent $\chi = 0.63 \pm 0.01$ corresponds to the known value $\chi \simeq 0.63$ [6]. We have also investigated here the scaling behaviour of the path closest to the origin and the scaling behaviour of the 'width' and the 'centre position' of the path's cluster—i.e. the scaling of the difference between the positions of the leftmost and the rightmost paths and the midpoint of these values. We observe that all these quantities scale with the same exponent, which also coincides, within our accuracy, with the directed-percolation exponent $\chi \simeq 0.63$.

Figure 3 also shows the scaling behaviour at $p = p_c$ of another important quantity, which characterizes the degeneracy of the best energy-paths cluster. It is the number of

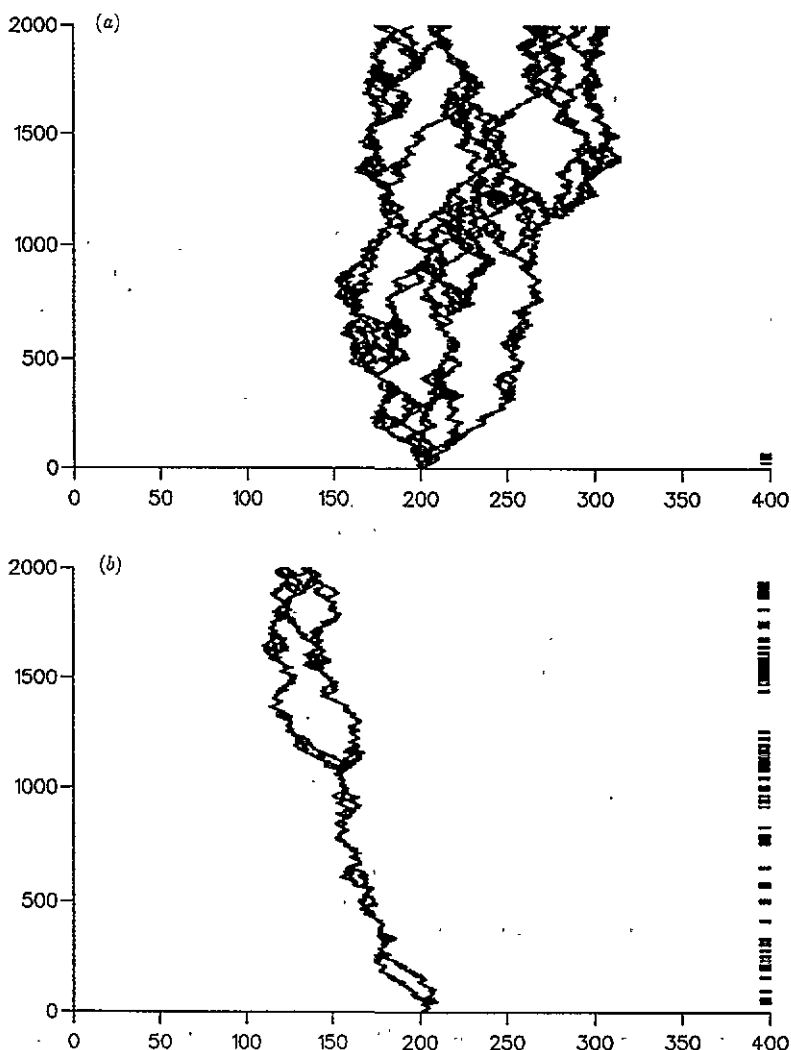


Figure 2. Typical shape of the cluster of best energy paths: (a) at the percolation threshold; (b) below the percolation threshold. Dashes on the right-hand side of the figure mark the values of t , at which the best energy path has to go on bonds with unit energy.

endpoints $n(t)$ of the paths of the cluster for fixed longitudinal distance t . The $n(t)$ at $p = p_c$ scales as $n(t) \sim t^\alpha$, with $\alpha \simeq 0.48 \pm 0.01$. The exponent α can be easily related to the exponent σ , characterizing the mass N of a finite cluster below percolation: $N \sim (p_c - p)^{-1/\sigma}$ [6]. Indeed, the infinite directed-percolation cluster at $p = p_c$ can be obtained as the limiting object of finite clusters as p approaches p_c from below. Thus, $N \sim t^{1+\alpha}$ should be proportional to $(\xi_{\parallel})^{1+\alpha}$, where $\xi_{\parallel} \sim (p_c - p)^{-\nu_{\parallel}}$ is the longitudinal size of a finite cluster. So for the exponent α , we should have $\alpha = 1/(\sigma \nu_{\parallel}) - 1 \simeq 0.47$ [6], in good agreement with our simulations. Proceeding further, following the lines of arguments of [6], we can also relate the exponent α to the fractal dimension of the directed-percolation cluster. However, the question of the fractal dimension of the percolation cluster is actually highly non-trivial. Indeed, it was shown recently [7] that, apart from the arguments given in [6], there are also two different ways to calculate the fractal dimension, which lead to

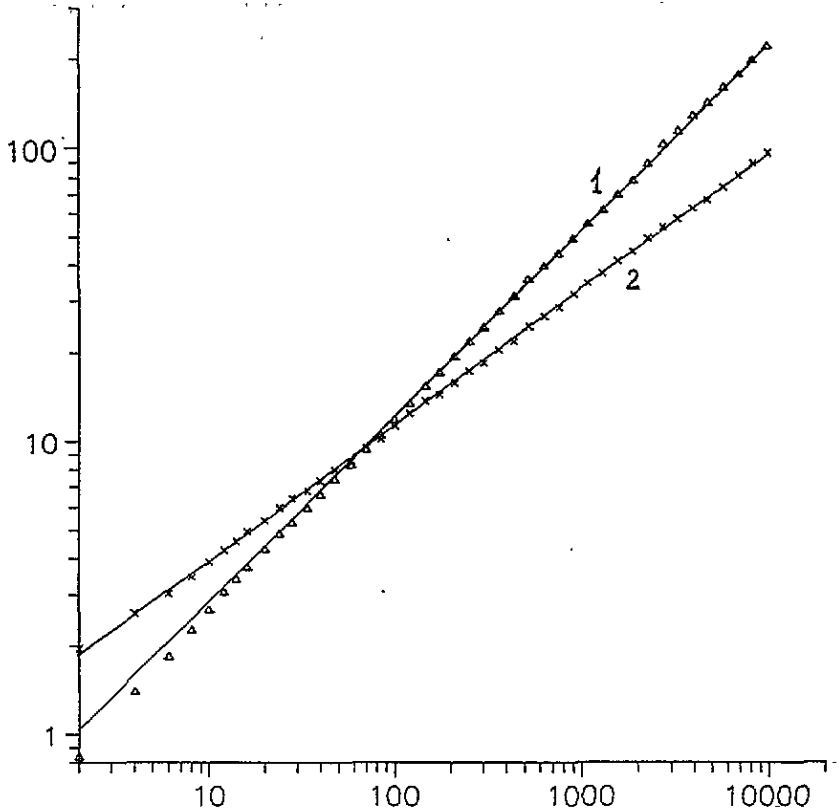


Figure 3. Line 1 shows the scaling of the position of the path, furthest away from the origin at $p = p_c$, with fitted exponent $\chi = 0.63 \pm 0.01$. Line 2 shows the scaling of the number of endpoints of the best energy paths for fixed longitudinal distance t at $p = p_c$, with $\alpha = 0.48 \pm 0.01$.

different results. At the same time, the careful box-counting method [7] gives a value which does not correspond to either of them. We will not touch the complicated question of the fractal dimension in this letter.

Now we describe our results for the best energy paths below the percolation threshold. A typical shape of these paths below the percolation is presented in figure 2(b). The scaling of the position of the path furthest from the origin for $p = 0.35$ is shown in figure 4. For comparison, we show on the same figure the test curve for the usual case of continuously-distributed random energy $V(x, t)$ (obtained from the same program). The difference in the exponents is small but evident ($\chi = 0.62 \pm 0.01$ and 0.67 ± 0.01 , respectively). We have also found that the position of the path closest to the origin and the transverse deviation weighted with the function $\varphi(x, t)$, which counts the total number of the best energy paths coming to a given point (x, t) , scale with the same exponent $\chi = 0.62 \pm 0.01$ (although in these cases we have some crossover at small t). So, within our accuracy, the scaling of the position of the best energy paths for $p < p_c$ is still the same as that at the percolation threshold ($\chi \simeq 0.63$). We have tested that this scaling holds in all of the region $p < p_c$, at least for p greater than 0.02. This gives us strong indication that $\chi \simeq 0.63$ is indeed the true asymptotic scaling law, and not a long intermediate regime due to the presence of some small parameter in the problem. At the same time, contrary to the case of $p = p_c$, the

number of endpoints of the best energy paths $n(t)$, for $p < p_c$ does not grow for $t \rightarrow \infty$. The asymptotic number $n(t \rightarrow \infty)$ grows for both limiting cases $p \rightarrow 0$ and $p \rightarrow p_c$.

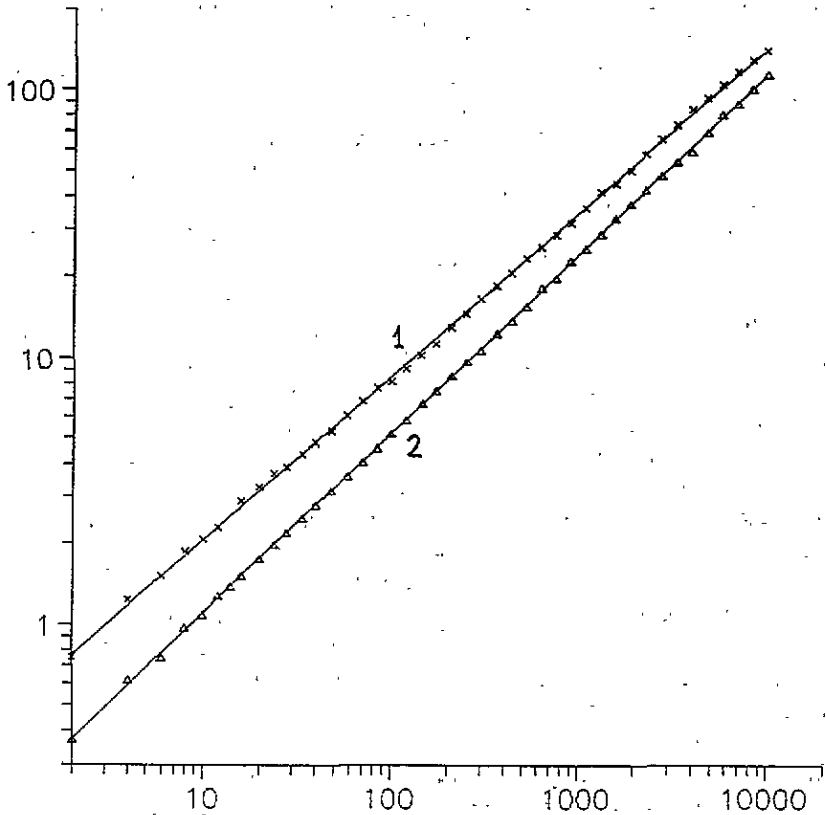


Figure 4. Line 1 shows the scaling of the position of the path, furthest away from the origin at $p = 0.35$, with $\chi = 0.62 \pm 0.01$. Line 2, shows for comparison, the test curve for the continuous distribution of the random potential, which gives exponent $\chi = 0.67 \pm 0.01$.

The scaling laws described above can be supported by some qualitative arguments. In [5], it was shown that the exponent $\chi = 0.63$ at $p = p_c$ is, in some sense, disorder independent: i.e. it is not changed if one assigns additional random energies to the bonds of the percolation cluster. However, the 'wrong' bonds, with energy unity connecting finite clusters of bonds with zero energy below p_c , can also be considered as a kind of disorder on the infinite percolation cluster. So, in this sense, one may expect the exponent $\chi \simeq 0.63$ to still hold at $p < p_c$ in correspondence with our numerical findings. A finite value for $n(t \rightarrow \infty)$ below p_c can be explained as follows. Below p_c , the number of 'wrong' bonds, which each path should pass, is proportional to the longitudinal distance t . Owing to fluctuations of concentration of the 'wrong' bonds, the typical energy difference between two paths increases with distance as the square root of t . So, one can expect that the ground state is globally unique, i.e. different best energy paths actually differ only locally (such loops are indeed seen in figure 2(b)). Thus, $n(t \rightarrow \infty)$ is finite, although it does not equal unity. With a decrease in concentration p , one has more and more regions with bonds with energy of unity only. Of course, inside such regions the best energy path is not uniquely

defined, so the degeneracy should be restored for $p \rightarrow 0$. Thus, the growth of $n(t \rightarrow \infty)$ for $p \rightarrow 0$ that we have found is also reasonable.

Up to now we have investigated only the properties of the best energy paths on a lattice, which corresponds to the ground state of the directed-polymer problem (1). The next important question is whether the directed-percolation exponent $\chi \simeq 0.63$ still holds at finite temperature T for the directed-polymer problem with discrete-valued disorder, or, at any $T \neq 0$, the usual directed polymer exponent is restored. Due to crossovers, which are stronger at finite T , and the small exponent difference ($\chi \simeq 0.63$ for the directed-percolation ground state and $\chi = \frac{2}{3}$ for the directed-polymer problem), it is not easy to find the answer to this question. However, one can investigate the difference in the exponents of the fluctuation of free energy, which may be much higher due to the contribution of entropy. Indeed, the entropy contribution, at least at $p = p_c$, should scale as $t^{1/2}$ [5]. We have searched for this contribution at $T \neq 0$ for $p < p_c$, but have not found it. The free-energy fluctuation always scales approximately as it should be for the conventional directed polymer: $\Delta F \sim t^\zeta$, $\zeta = 0.33 \pm 0.05$ (ζ is exactly $\frac{1}{3}$ for the usual directed polymer). Although the uncertainty in the exponent is now much larger than in the calculations for the ground-state properties, the possibility of $\zeta = \frac{1}{2}$ can be definitely ruled out. We consider this fact as an indication that the directed-percolation ground state at $p < p_c$ is unstable.

The last question which we shall address in this letter is the scaling of the best energy paths for a random potential taking $q > 2$ different values (with equal probabilities). Evidently, for large enough q , one should reproduce the result for continuous energy distribution. Indeed, for $q \geq 12$, the scaling dependence for the position of the path furthest from the origin is indistinguishable from the test curve for continuous distribution. For $q = 3$, we still have the directed-percolation exponent. However, due to the small exponent difference, it is difficult to determine precisely the transition value of q . Most probably it equals 5 or 6.

In summary, we have investigated the ground-state properties (the limit of zero temperature T) for the special case of the directed-polymer problem when the random potential takes only two discrete values. In the entire region below the directed-percolation threshold ($p < p_c$), the ground state retains a scaling exponent which is identical, with numerical accuracy, to the directed-percolation exponent ($\chi \simeq 0.63$). Although we have found some indication that this directed percolation ground state is unstable below the percolation threshold, the question of stability needs further detailed study.

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